MTH 403: Real Analysis II

Practice Assignment I

Differentiation

1. Examine the differentiability of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ assuming that $f((0,0) = 0$.

   (a) $f(x,y) = \dfrac{x^3}{x^2+y^2}$, $(x,y) \neq (0,0)$.

   (b) $f(x,y) = \sqrt{|xy|}$, $(x,y) \neq (0,0)$.

   (c) $f(x,y) = \dfrac{x|y|}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$.

   (d) $f(v) = |v| \cdot g\left(\dfrac{v}{|v|}\right)$, where $v = (x,y) \in \mathbb{R}^2$ and g is a continuous real-valued odd function on the unit circle such that $g(0,1) = g(1,0) = 0$.

2. Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is a function such that $|f(x)| \leq |x|^2$, then f is differentiable at zero.

3. Determine whether the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ are of class C^2.

   (a) $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

   (b) $$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

4. Compute the derivative (matrix) of the following functions.

   (a) $f(x,y,z) = \sin(x\sin(y\sin z))$

   (b) $f(x,y,z) = (x+y)^z$

   (c) $f(x,y) = \int_a^x g$, where $g : \mathbb{R} \to \mathbb{R}$ is continuous.

   (d) $f(x,y) = \int_y^x g$, where $g : \mathbb{R} \to \mathbb{R}$ is continuous.

5. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

   is a C^∞ function.

6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x,y) = f(x,y,g(x,y))$.

(a) Find DF in terms of $D_i f$ and $D_i g$.

(b) If $F(x, y) = 0$ for all (x, y), find $D_i g$ in terms of $D_i f$.

7. Using the Inverse Function Theorem (whenever applicable), verify where the function f has an inverse (say g) in a neighborhood of the indicated point. If g exists, then also compute Dg at the given point.

   (a) $f(x, y) = (e^x \cos y, e^x \sin y)$ at $(0, \pi)$.

   (b) $f(x, y) = (x^2 - y^2, 2xy)$ at $(0, 1)$.

   (c) $f : \mathbb{R}^n \to \mathbb{R}^n : x \overset{f}{\mapsto} |x| \cdot x$ at 0.

8. Show that a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$, where $m < n$, is not injective.

9. For $A \subset \mathbb{R}^n$ open, let $f : A \to \mathbb{R}^n$ be a C^r function such that $Df(x) \neq 0$, for all $x \in A$. Show that $f(A)$ is a always open.

10. Suppose that $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ is of class C^1 such that $f(a) = 0$ and $Df(a)$ is of rank n for some $a \in \mathbb{R}^{k+n}$. Show that for a point $c \in \mathbb{R}^n$ sufficiently close to zero, $f(x) = c$ has a solution.

11. Let $f : \mathbb{R}^5 \to \mathbb{R}^2$ be of class C^1 such that

$$Df(a) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix} \text{ at } a = (1, 2, -1, 3, 0).$$

   Show that there exists a $g : B \to \mathbb{R}^2$ (B open) of class C^1 such that

$$f(x_1, g_1(x), g_2(x), x_2, x_3) = 0,$$

   for $x = (x_1, x_2, x_3) \in B$ and $g(1, 3, 0) = (2, -1)$.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be of class C^2 with $f(0, 0) = 0$ and $Df(0, 0) = [2\ 3]$. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$g(x, y, z) = f(x + 2y + 3z - 1, x^3 + y^2 - z^2).$$

   Show that $g(x, y, z) = 0$ is solvable as $z = h(x, y)$ in some neighborhood of $(-2, 3)$ such that $h(-2, 3) = -1$.

13. Reading assignment: Read Lemma 2.10 and Theorem 2.13 from Calculus on manifolds by M. Spivak.